Optimal Control of Investment for an Insurer in Two Currency Markets

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Abstract In this paper, we study the optimal investment problem of an insurer whose surplus process follows the diffusion approximation of the classical Cramer-Lundberg model. Investment in the foreign market is allowed, and therefore, the foreign exchange rate model is considered and incorporated. It is assumed that the instantaneous mean growth rate of foreign exchange rate price follows an Ornstein-Uhlenbeck process. Dynamic programming method is employed to study the problem of maximizing the expected exponential utility of terminal wealth. By solving the corresponding Hamilton-Jacobi-Bellman equations, the optimal investment strategies and the value functions are obtained. Finally, numerical analysis is presented.

Keywords: Cramer-Lundberg model; Exponential utility; Hamilton-Jacobi-Bellman equation; Optimal investment strategy; Foreign exchange rate

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1 Introduction

In actuarial science and applied probability, risk theory is a traditional and modern field, which uses mathematical models to describe an insurer's vulnerability to ruin. In order to decrease (increase) the risk (profit), the insurance companies are allowed to invest their wealth into risk-free assets and risky assets. And in recent years, there are many remarkable works of optimal investment problems. Especially, maximizing the expected exponential utility function and minimizing the probability of ruin have attracted a substantial amount of interest.

Browne [5] used Brownian motion with drift to describe the surplus of the insurance company and found the optimal investment strategy to maximize the expected exponential utility of the terminal wealth and minimize the probability of ruin. Later, Yang and Zhang [22] explored the same optimal investment problem for a risk process modeled by a jump diffusion process. In Hipp and Pulm [12], the authors considered a risk process modeled as a compound Poisson process and investigated the optimal investment strategy to minimize

the ruin probability of this model. Liu and Yang [15] generalized the model in Hipp and Pulm [12] by including a risk-free asset. And then the optimal investment strategy was investigated. Schmidli [19] also studied the compound Poisson risk model and the optimal investment strategy of minimizing its ruin probability. In Bai and Guo [4], they considered the optimal problem with multiple risky assets and no-shorting constraint. By solving the corresponding Hamilton-Jacobi-Bellman equations, the optimal strategies for maximizing the expected exponential utility and minimizing the ruin probability were obtained. Wang [21] considered the optimal investment strategy to maximize the exponential utility of an insurance company's reserve. The claim process was supposed to be a pure jump process, which is not necessarily compound Poisson, and the insurer has the option of investing in multiple risky assets.

However, prior studies did not consider the condition that the insurers are allowed to invest their wealth in more than one currency market. Thus, in this paper we investigate the case that the insurance company is allowed to invest its wealth into more than one currency market, such that it can invest its wealth into domestic risk-free assets and foreign risky assets.

The connection between the domestic currency market and the foreign currency market is the exchange rate. Foreign exchange rate plays an important role as a tool used to convert foreign market cash flows into domestic currency. And there are many factors that influence exchange rate prices, such as inflation, balance of international payment, interest-rate spread, etc.

Inflation is the most important fundamental factor affecting the movements of exchange rate. If the inflation rate of a domestic country is higher than that of a foreign country, the competitiveness of the domestic country's exports is weakened, while this increases the competitiveness of foreign goods in domestic country's market. It would cause the domestic country's trade balance of payments deficit and the demand of foreign exchange is larger than the supply. Then it leads to the increase of foreign exchange rate price. Conversely, the price of foreign exchange rate declines.

The balance of payments is the direct factor that affects the exchange rate. For example, when a country has a large balance of payments surplus, i.e., the country's imports are less than its exports, its currency demand will increase, which will lead to an increase in foreign exchange flowing into the country. In this way, in the foreign exchange market, the supply of foreign exchange is greater than the demand then the foreign exchange rate price goes down. But if a country has a large balance of payments deficit, i.e., the country's imports are more than its exports, the supply of foreign exchange is greater than the demand which leads to the increase of foreign exchange rate price.

Under certain conditions, interest rates have a great short-term impact on exchange rate price. This effect is caused by the difference of interest rates between different countries. In general, if interest- rate spread is increasing then the demand for domestic currency increases. This leads to the increase of the domestic currency price. And thus the foreign exchange rate price declines. Conversely, if the interest- rate spread goes down, the price of foreign exchange rate is increasing.

As we all known, the classical geometric Brownian motion is the most commonly used model to describe the dynamics of exchange rate price. In Musiela and Rutkowski [16], the explicit valuation formulas for various kinds of currency and foreign equity options were established in which the foreign exchange rate was modeled by means of geometric

Brownian motion. Veraart [20] considered an investor in the foreign exchange market who trades in domestic currency market and foreign currency market with the exchange rate modeled as a geometric Brownian motion. After that, a more general model of exchange rate was used. In Eisenberg [6], the author considered an insurance company seeking to maximize the expected discounted dividends whereas the dividends are declared or paid in a foreign currency. It was assumed that the insurance company generates its income in a foreign currency but pays dividends in its home currency. In that paper, the exchange rate was modeled by a geometric Lévy process. However, taking into account of a variety of factors which affect the exchange rate, the classical geometric Brownian motion can not better reflect the real dynamics of exchange rate. Thus other models have been created to describe exchange rate. One of the most popular model is the one in which the interest-rate spread is incorporated into the geometric Brownian motion. For example, in Guo, et al. [11], the authors investigated the optimal strategy of an insurer who invests in both domestic and foreign markets. They assumed that the domestic and foreign nominal interest rates are both described by extended Cox-Ingersoll-Ross (CIR) model. And the exchange rate price is modeled by geometric Brownian motion with domestic and foreign interest rates. For more details of the model of exchange rate price described by geometric Brownian motion with interest rates one can refer [1–3].

Although interest-rate spread has a certain impact on exchange rate price and the models with interest-rate spread are excellent, from the perspective of the basic factors determining the trend of exchange rate fluctuation that the effect of interest- rate spread is limited. Moreover, from the above descriptions we obtain that the supply and demand of domestic and foreign currencies is the most paramount and direct factor on the change of exchange rate price. Thus in our paper, the foreign exchange market's supply and demand is incorporated into the model of exchange rate price.

In Liang, et al. [14] and Rishel [17], they used the model of geometric Brownian motion, where the mean growth rate is given by Ornstein-Uhlenbeck process, to describe the dynamics of risky asset price which can have features of bull and bear markets. Inspired by the models of risky assets in Liang, et al. [14] and Rishel [17] and the discussions of the effect factors of foreign exchange rate, we consider that the foreign exchange rate also has the bull and bear markets. That is when the demand of foreign currency is larger than the supply, the foreign exchange rate price increases, we call it the "bull foreign exchange rate". In addition, when the demand of foreign currency is less than the supply, the foreign exchange rate price goes down, we call it the "bear foreign exchange rate". The commonly-used model for the exchange rate price is the geometric Brownian motion in which the expected instantaneous rate and the volatility of the exchange rate price are both constants. This seems to rule out bull and bear markets.

In this paper, the price of foreign exchange rate Q_t is described by the following differential equation:

$$dQ_t = Q_t\{a(t)dt + \sigma_Q dW_t^2\}, \; Q_0 = q, \qquad (1.1)$$

where $a(t) = u_Q + m(t)$ and $m(t)$ is given by the Ornstein-Uhlenbeck equation

$$dm(t) = \alpha m(t)dt + \beta dW_t^3, \; m(0) = m_0.$$

Here $u_Q, \sigma_Q, q, \alpha, \beta$ are known constants and they are all positive except α and β. In (1.1), u_Q is the target mean growth rate for the exchange rate. If $m(t) > 0$, then $a(t)$ is

substantially larger than u_Q, this could be considered as a bull market of exchange rate. Conversely, if $m(t) < 0$ then $a(t)$ is substantially less than u_Q, this could be considered as a bear market of exchange rate. Especially, if $a(t) < 0$ the exchange rate price goes down. Here the function of $m(t)$ is to let the random mean growth rate of foreign exchange rate price be close to the target mean growth rate. Once the mean growth rate of foreign exchange rate price is larger than the target mean growth rate for a long time, then $m(t) < 0$. Otherwise, $m(t) > 0$.

The insurance company is allowed to invest its wealth into domestic and foreign currency markets with the exchange rate price Q_t described by (1.1). Our target is to maximize the expected exponential utility of terminal wealth over all admissible strategies.

The rest of the paper is organized as follows. The model is described in Section 2. Our main results are given in Section 3. By solving the corresponding Hamilton-Jacobi-Bellman equations, the optimal value functions and optimal strategies are explicitly derived. In particular, we find that if the insurance company only invests in foreign risky assets and the price of exchange rate is modeled by geometric Brownian motion then the optimal investment strategy is a constant, regardless of the level of wealth the company has. In the last section, numerical examples and analysis are presented. And we find that, in some cases, investing into two currency markets can produce a higher value function than investing into only one currency market.

2 The model

We start from the classical Cramer-Lundberg model in which the surplus of the insurance company is modeled as

$$X_t = X_0 + pt - \sum_{i=1}^{N(t)} Z_i, \text{ with } X_0 = x,$$

where p is the premium rate, $N(t)$ is the Poisson process stating the number of claims, and Z_i is a sequence of independent random variables which are identically distributed representing the size of claims. Without the loss of generality we assume that the intensity of the process $N(t)$ is 1, then the dynamics of X_t can be approximated by

$$dX_t = udt + \sigma dW_t, \quad X_0 = x,$$

where $u = p - E[Z] > 0, \sigma^2 = E[Z^2]$, and W_t is a standard Brownian motion. For more details of the diffusion approximation of the surplus process, we can refer to [7, 9, 13, 18].

The insurance company invests its wealth into domestic risk-free asset and foreign risky asset. The price of domestic risk-free asset is given by

$$dB_t^d = B_t^d r_d dt, \quad B^d(0) = B_0^d.$$

The foreign risky asset price S_t^f is modeled by means of geometric Brownian motion such that

$$dS_t^f = S_t^f(u_f dt + \sigma_f dW_t^1), \quad S^f(0) = S_0^f. \tag{2.2}$$

We adopt here the convention that the price S_t^f is denominated by foreign currency and the exchange rate is denominated in units of domestic currency per unit of foreign currency. This means that Q_t represents the domestic price at time t of one unit of the foreign currency. Thus let $g_t := g(S_t^f, Q_t) = Q_t S_t^f$, then g_t is the price of foreign risky asset denominated by domestic currency. By Itǒ's formula and the formulas of (1.1) and (2.2) we find that g_t satisfies the following stochastic differential equation

$$dg_t = g_t\{(u_f + a(t))dt + \sigma_f dW_t^1 + \sigma_Q dW_t^2\}.$$

The total amount of money invested in foreign risky asset at time t is denoted by π_t and the rest of the surplus is invested into domestic risk-free asset. Under the strategy π_t, the surplus of the insurance company is as follows:

$$dX_t^\pi = \{\pi_t A_1 + u + r_d X_t + \pi_t m(t)\}dt + \sigma dW_t + \pi_t \sigma_f dW_t^1 + \pi_t \sigma_Q dW_t^2, \qquad (2.3)$$

where $A_1 = u_f + u_Q - r_d$ and the initial surplus is $X_0^\pi = x_0$. Here it is allowed that $\pi_t < 0$ and $\pi_t > X_t^\pi$ which means that the company is allowed to short sell the foreign risky asset and borrow money for investment in foreign risky asset. And we assume that W_t, W_t^1, W_t^2, W_t^3 are independent standard Brownian motions on the same probability space $(\Omega, \mathcal{F}, P)$.

We are going to maximize the expected exponential utility of terminal wealth over all admissible strategies π_t. A strategy π_t is said to be admissible, if $\pi.$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the filtration generated by X_t^π, and for any $T > 0$, $E[\int_0^T \pi^2(t)dt] < \infty$. The set of all admissible strategies is denoted by Π.

3 The main results

Suppose now that the insurance company is interested in maximizing the utility of its terminal wealth at time T. Denote the utility function as $u(x)$ with $u'(x) > 0$ and $u''(x) < 0$. For a strategy π, the utility attained by the insurer from state x, m at time t is defined as

$$V_\pi(t, x, m) = E[u(X_T^\pi)|(X_t^\pi, m(t)) = (x, m)].$$

Our objective is to find the optimal value function

$$V(t, x, m) = \sup_{\pi \in \Pi} V_\pi(t, x, m) \qquad (3.4)$$

and the optimal investment strategy π^* such that $V_{\pi^*}(t, x, m) = V(t, x, m)$.

Assume now that the investor has an exponential utility function

$$u(x) = \lambda - \frac{\gamma}{\theta} e^{-\theta x}, \qquad (3.5)$$

where $\gamma > 0$ and $\theta > 0$. The utility function (3.5) plays a remarkable part in insurance mathematics and actuarial practice, since it is the only utility function under which the principle of "zero utility" gives a fair premium that is independent of the level of reserve of an insurance company (see Gerber [10]).

Applying the dynamic programming approach described in [8], from standard arguments, we see that if the optimal value function $V(t, x, m)$ and its partial derivatives $V_t, V_x, V_{xx}, V_m, V_{mm}$ are continuous on $[0, T] \times R^1 \times R^1$, then $V(t, x, m)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$
\begin{aligned}
V_t &+ \sup_\pi \left\{ [\pi A_1 + \pi m + x r_d + u] V_x + \frac{1}{2} [\sigma^2 + \pi^2 (\sigma_f^2 + \sigma_Q^2)] V_{xx} \right\} \\
&+ \alpha m V_m + \frac{1}{2} \beta^2 V_{mm} = 0,
\end{aligned}
\tag{3.6}
$$

with boundary condition $V(T, x, m) = u(x)$.

In order to solve the HJB equation (3.6), we first find the value $\pi(x, m)$ which maximizes the function

$$
\left(\pi A_1 + \pi m + x r_d + u \right) V_x + \frac{1}{2} [\sigma^2 + \pi^2 (\sigma_f^2 + \sigma_Q^2)] V_{xx}
\tag{3.7}
$$

Differentiating with respect to π in (3.7) the optimizer

$$
\pi^* = -\frac{A_1 + m}{\sigma_f^2 + \sigma_Q^2} \frac{V_x}{V_{xx}}
\tag{3.8}
$$

is obtained.

Assume that HJB equation (3.6) has a classical solution V such that $V_x > 0$ and $V_{xx} < 0$. Inspired by the form of the solution in [5], we try to find the solution of (3.6) as the form

$$
V(t, x, m) = \lambda - \frac{\gamma}{\theta} exp \left\{ -\theta x e^{r_d(T-t)} + h(t, m) \right\},
\tag{3.9}
$$

where $h(t, m)$ is a suitable function such that (3.9) is a solution of (3.6). And the boundary condition $V(T, x, m) = u(x)$ implies that $h(T, m) = 0$.

From (3.9) we can calculate that

$$
V_t = \left[V(t, x, m) - \lambda \right] \left\{ \theta x r_d e^{r_d(T-t)} + h_t \right\}
$$
$$
V_x = -\left[V(t, x, m) - \lambda \right] \theta e^{r_d(T-t)}, \quad V_{xx} = \left[V(t, x, m) - \lambda \right] \theta^2 e^{2r_d(T-t)}
$$
$$
V_m = \left[V(t, x, m) - \lambda \right] h_m, \quad V_{mm} = \left[V(t, x, m) - \lambda \right] (h_m^2 + h_{mm}),
$$

where $V_t, V_x, V_{xx}, V_m, V_{mm}$ are the partial derivatives of $V(t, x, m)$ and h_t, h_m, h_{mm} are the partial derivatives of $h(t, m)$. Substituting $V_t, V_x, V_{xx}, V_m, V_{mm}$ back into (3.6) yields

$$
\begin{aligned}
h_t &+ \sup_\pi \left\{ -\pi (A_1 + m) \theta e^{r_d(T-t)} - u \theta e^{r_d(T-t)} + \frac{1}{2} \pi^2 \theta^2 (\sigma_f^2 + \sigma_Q^2) e^{2r_d(T-t)} \right\} \\
&+ \frac{1}{2} \theta^2 \sigma^2 e^{2r_d(T-t)} + \alpha m h_m + \frac{1}{2} \beta^2 (h_m^2 + h_{mm}) = 0.
\end{aligned}
\tag{3.10}
$$

And from (3.8)

$$
\pi^* = \frac{A_1 + m}{\theta(\sigma_f^2 + \sigma_Q^2)} e^{-r_d(T-t)}.
\tag{3.11}
$$

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Put π^* into (3.10) and calculate then

$$h_t - u\theta e^{r_d(T-t)} + \frac{1}{2}\theta^2\sigma^2 e^{2r_d(T-t)} - \frac{1}{2}\frac{(A_1+m)^2}{\sigma_f^2 + \sigma_Q^2} + \alpha m h_m + \frac{1}{2}\beta^2(h_m^2 + h_{mm}) = 0. \quad (3.12)$$

It can be shown that (3.9) is a solution to (3.10) if $h(t, m)$ is a solution to (3.12).

Theorem 3.1. *With the terminal condition $h(T, m) = 0$, the partial differential equation (3.12) has the solution of the form*

$$h(t, m) = K(t)m^2 + L(t)m + J(t), \quad (3.13)$$

where $K(t)$ is a solution to

$$K'(t) + 2\beta^2 K^2(t) + 2\alpha K(t) - \frac{1}{2(\sigma_f^2 + \sigma_Q^2)} = 0, \quad K(T) = 0; \quad (3.14)$$

$L(t)$ is a solution to

$$L'(t) + (\alpha + 2\beta^2 K(t))L(t) - \frac{A_1}{\sigma_f^2 + \sigma_Q^2} = 0, \quad L(T) = 0; \quad (3.15)$$

and $J(t)$ is a solution to

$$J'(t) - u\theta e^{r_d(T-t)} + \frac{1}{2}\theta^2\sigma^2 e^{2r_d(T-t)} - \frac{A_1^2}{2(\sigma_f^2 + \sigma_Q^2)} + \frac{1}{2}\beta^2 L^2 + \beta^2 K = 0, \quad J(T) = 0. \quad (3.16)$$

Proof. Substituting (3.13) into (3.12) and combining like terms with respect to the powers of m, we have that

$$m^2\left\{K'(t) + 2\beta^2 K^2(t) + 2\alpha K(t) - \frac{1}{2(\sigma_f^2 + \sigma_Q^2)}\right\} +$$

$$m\left\{L'(t) + \alpha L(t) + 2\beta^2 K(t)L(t) - \frac{A_1}{\sigma_f^2 + \sigma_Q^2}\right\} + \left\{J'(t)\right.$$

$$\left. -u\theta e^{r_d(T-t)} + \frac{1}{2}\theta^2\sigma^2 e^{2r_d(T-t)} - \frac{A_1^2}{2(\sigma_f^2 + \sigma_Q^2)} + \frac{1}{2}\beta^2 L^2(t) + \beta^2 K(t)\right\} = 0. \quad (3.17)$$

Then it is obvious that (3.13) is a solution to (3.12) if $K(t), L(t), J(t)$ are solutions to the differential equations (3.14), (3.15) and (3.16), respectively.

$\square$

Then we are going to solve the differential equations (3.14), (3.15) and (3.16), respectively.

Let

$$B := 2\beta^2, C := 2\alpha, D := -\frac{1}{2(\sigma_f^2 + \sigma_Q^2)},$$

then the Riccati equation (3.14) becomes

$$K'(t) + BK^2(t) + CK(t) + D = 0, \quad K(T) = 0. \quad (3.18)$$

If $B \neq 0$, i.e. $\beta \neq 0$, integrating

$$\frac{dK(t)}{BK^2(t) + CK(t) + D} = -dt$$

on both sides with respect to t we obtain that

$$\int \frac{dK(t)}{BK^2(t) + CK(t) + D} = -t + E, \tag{3.19}$$

where E is a constant. Since $\Delta = C^2 - 4BD = 4\alpha^2 + \frac{4\beta^2}{\sigma_f^2 + \sigma_Q^2} > 0$, the quadratic equation $BK^2(t) + CK(t) + D = 0$ has two different real roots given by

$$K_1, K_2 = \frac{-C \pm \sqrt{C^2 - 4BD}}{2B}. \tag{3.20}$$

Substituting (3.20) into (3.19) and considering the boundary condition $K(T) = 0$ then we obtain

$$K(t) = \frac{K_1 - K_2 e^{B(K_1 - K_2)(t-T)}}{1 - (K_1/K_2)e^{B(K_1 - K_2)(t-T)}}. \tag{3.21}$$

If $B = 0$, i.e. $\beta = 0$, then

$$K(t) = \frac{1}{4\alpha(\sigma_f^2 + \sigma_Q^2)} - \frac{1}{4\alpha(\sigma_f^2 + \sigma_Q^2)}e^{2\alpha(T-t)}. \tag{3.22}$$

With the value of $K(t)$ defined in (3.21) or (3.22), the linear ordinary equation (3.15) has the solution of the form

$$L(t) = e^{\int_t^T (\alpha + 2\beta^2 K(s))ds}\left[\int_t^T -\frac{A_1}{\sigma_f^2 + \sigma_Q^2}e^{\int_t^T -(\alpha + 2\beta^2 K(y))dy}ds\right]. \tag{3.23}$$

And the solution of (3.16) is given by

$$\begin{aligned} J(t) &= \frac{u\theta}{r_d}(1 - e^{r_d(T-t)}) - \frac{\theta^2\sigma^2}{4r_d}(1 - e^{2r_d(T-t)}) - \frac{A_1^2}{2(\sigma_f^2 + \sigma_Q^2)}(T - t) \\ &+ \int_t^T (\frac{1}{2}\beta^2 L^2(s) + \beta^2 K(s))ds. \end{aligned} \tag{3.24}$$

From [8] the following verification theorem exists.

Theorem 3.2. *Let $W \in C^{1,2}([0,T] \times R^2)$ be a classical solution to the HJB equation (3.6) with the boundary condition $W(T, x, m) = u(x)$, then the value function V given by (3.4) coincides with W such that*

$$W(t, x, m) = V(t, x, m).$$

In addition, let π^* be the optimizer of (3.6), that is for any $(t, x, m) \in [0, T] \times R^2$

$$V_t + \left[\pi^*(A_1 + m) + xr_d + u\right]V_x + \frac{1}{2}\left[\sigma^2 + \pi^{*2}(\sigma_f^2 + \sigma_Q^2)\right]V_{xx} + \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} = 0.$$

Then $\pi^*(t, X_t^*, m(t))$ is the optimal strategy with

$$V_{\pi^*}(t, x, m) = V(t, x, m),$$

where X_t^* is the surplus process under the optimal strategy π^*.

From the above statements we have the following results.

Theorem 3.3. *With the utility function (3.5), the optimal strategy for the optimization problem (3.4) subject to (2.3) is*

$$\pi_t^* = \frac{A_1 + m(t)}{\theta(\sigma_f^2 + \sigma_Q^2)}e^{-r_d(T-t)}, \quad \forall t \in [0, T].$$

And the value function is given by the form

$$V(t, x, m) = \lambda - \frac{\gamma}{\theta}exp\Big\{ - \theta x e^{r_d(T-t)} + h(t, m)\Big\}$$

with $h(t, m) = K(t)m^2 + L(t)m + J(t)$, *where* $K(t), L(t)$ *and* $J(t)$ *are given by (3.21)-(3.24).*

If $m(t) = 0$ in (1.1) then the exchange rate price $Q(t)$ is degenerated into the process which is modeled by means of geometric Brownian motion

$$dQ(t) = Q(t)(u_Q dt + \sigma_Q dW_t^2).$$

In this case, under the control of π, X_t^π satisfies the following stochastic equation

$$dX_t^\pi = \big\{\pi_t A_1 + u + r_d X_t\big\}dt + \sigma dW_t + \pi_t \sigma_f dW_t^1 + \pi_t \sigma_Q dW_t^2. \tag{3.25}$$

Then the HJB equation in (3.6) becomes to be

$$V_t + \sup_\pi \Big\{ \big[\pi A_1 + x r_d + u\big]V_x + \frac{1}{2}\big[\sigma^2 + \pi^2(\sigma_f^2 + \sigma_Q^2)\big]V_{xx}\Big\} = 0. \tag{3.26}$$

By solving the above HJB equation (3.26) the following corollary is obtained.

Corollary 3.4. *With the* X_t^π *in (3.25) the optimal investment strategy is given by*

$$\pi_t^* = \frac{A_1}{\theta(\sigma_f^2 + \sigma_Q^2)}e^{-r_d(T-t)}, \quad \forall t \in [0, T].$$

Furthermore, the value function has the form

$$V(t, x) = \lambda - \frac{\gamma}{\theta}exp\Big\{ - \theta x e^{r_d(T-t)} + f(T - t)\Big\}$$

where

$$f(T - t) = \frac{\theta u}{r_d}(1 - e^{r_d(T-t)}) - \frac{\theta^2\sigma^2}{4r_d}(1 - e^{2r_d(T-t)}) - \frac{A_1^2}{2(\sigma_f^2 + \sigma_Q^2)}(T - t).$$

Let S_t^d be the price of domestic risky asset described by the following stochastic differential equation

$$dS_t^d = S_t^d(u_d dt + \sigma_d dW_t^4), \tag{3.27}$$

where u_d and σ_d are positive constants and W_t^4 is the standard Brownian motion which is independent of W_t. Assume the insure invests his wealth only in domestic currency market, i.e., domestic risk-free asset and domestic risky asset, then the surplus under the control π is that

$$dX_t^\pi = \big\{\pi(u_d - r_d) + u + r_d X_t\big\}dt + \sigma dW_t + \pi\sigma_d dW_t^4. \tag{3.28}$$

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Corollary 3.5. *In the domestic currency market, the optimal strategy for the optimization problem (3.4) is*

$$\pi_t^* = \frac{u_d - r_d}{\theta \sigma_d^2} e^{-r_d(T-t)}, \quad \forall t \in [0, T],$$

and the corresponding value function has the form

$$V(t, x) = \lambda - \frac{\gamma}{\theta} exp\{ - \theta x e^{r_d(T-t)} + g(T - t)\}$$

where

$$g(T - t) = \frac{\theta u}{r_d}(1 - e^{r_d(T-t)}) - \frac{\theta^2 \sigma^2}{4r_d}(1 - e^{2r_d(T-t)}) - \frac{(u_d - r_d)^2}{2\sigma_d^2}(T - t).$$

Remark 3.6. *By comparing the optimal investment strategies and value functions in Corollary 3.4 and Corollary 3.5, it is not difficult to see that*

(i) Suppose the insurer invests the same amount of his wealth into domestic risky assets and foreign risky assets. We can see that

If $u_f + u_Q \geq u_d$, then the value function with exchange rate is always larger than the value function without exchange rate. Conversely, if $u_f + u_Q < u_d$ it is better for the insurer to invest in domestic risky assets.

(ii) Assume that the insurer wants to get the same value functions in the two kinds of currency markets.

If $u_f + u_Q \geq u_d$, in order to get the same value functions in the two cases, the amount of wealth invested in foreign risky assets are less than that invested in domestic risky assets. In addition, if $u_f + u_Q < u_d$, in order to get the same values of the value functions the insurer should invest more in foreign risky assets.

(iii) When $u_f + u_Q \geq u_d$ if the insurer invests more in foreign risky assets the value function is higher than the value function in domestic risky assets.

Remark 3.7. *From Corollaries 3.4 and 3.5, it is not difficult to see that if no domestic risk-free assets are traded and only risky asset is considered even in domestic or foreign market, the optimal strategies are always constants, regardless of the level of wealth the insurer has.*

4　Numerical examples and analysis

In order to demonstrate our results, numerical examples are presented for the optimal investment strategies and value functions in two kinds of currency markets. Our objective is to study the effect of exchange rate on the insurer's decision and the value function. The particular numbers of basic parameters are given in the following tables.

T	r_d	λ	θ	γ	u	σ	u_f	σ_f	u_Q	σ_Q	x	u_d	σ_d
4	0.1	1	1	1	0.4	0.1	0.3	$\sqrt{0.1}$	0.2	$\sqrt{0.3}$	2	0.3	$\sqrt{0.2}$

Table 1

(a)   (b)
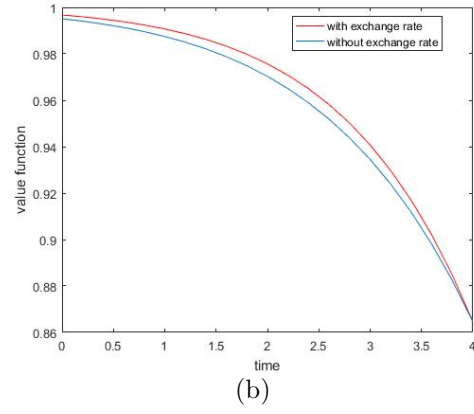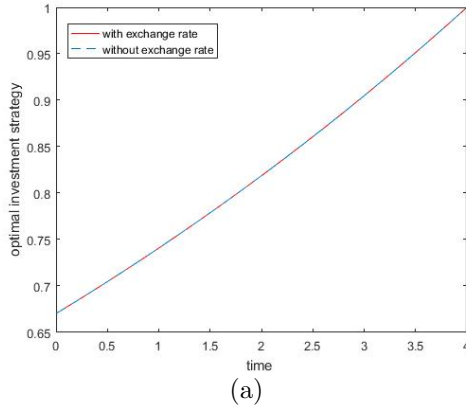
The numbers in Table 1 satisfy that

$$\sigma_d^2(u_f + u_Q - r_d) = 0.08 = (\sigma_f^2 + \sigma_Q^2)(u_d - r_d)$$

and

$$u_f + u_Q = 0.5 > u_d = 0.3.$$

The graphs of the optimal investment strategies and value functions corresponding to the data in Table 1 are shown in (a) and (b). They show that if the insure invests the same amount of its wealth into foreign and domestic risky assets, then the former produces a larger value function. Thus it is better for the insurer to invest in foreign risky assets. And the results in graphs (a) and (b) also coincide with the conclusions in (i) of Remark 3.6.

T	r_d	λ	θ	γ	u	σ	u_f	σ_f	u_Q	σ_Q	x	u_d	σ_d
4	0.1	1	1	1	0.4	0.1	0.3	0.2	0.2	$\sqrt{0.12}$	2	0.3	0.2

Table 2

(c)   (d)
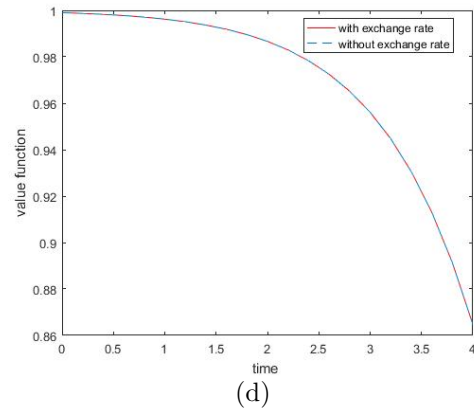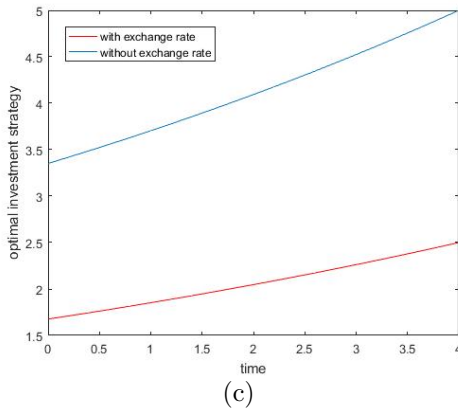
By employing the numbers in Table 2, the graphs of optimal investment strategies and value functions are given in (c) and (d). From graphs (c) and (d) it is not difficult

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to see that if the insurer wants to get the same value functions in the currency markets with and without exchange rates, he should invest more in domestic risky assets than in foreign risky assets. And the numbers in Table 2 satisfy that

$$\sigma_d^2(u_f + u_Q - r_d)^2 = (\sigma_f^2 + \sigma_Q^2)(u_d - r_d)$$

and

$$u_f + u_Q > u_d.$$

Thus graphs (c) and (d) reflect the results in (ii) of Remark 3.6.

The graphs (e) and (f) are obtained from the numbers listed in Table 3. We first study the effect of the exchange rate on optimal investment strategies. When the insurer has exponential preferences, the realization of their optimal investment strategies are illustrated in Figure (e). It can be seen that, when there are two currency markets the insurer invests a larger proportion of her wealth in the risky asset.

Secondly, we explore the effect of the exchange rate on the value function. From figure (f), we can easily find that, it is much better for the insurer to invest her surplus in foreign risky asset to decrease the risk. The value function with exchange rate is always larger than the value function without exchange rate, except the terminal value. It indicates that it is better to incorporate the exchange rate in the model.

T	r_d	λ	θ	γ	u	σ	u_f	σ_f	u_Q	σ_Q	x	u_d	σ_d
4	0.1	1	1	1	0.4	0.1	0.2	0.3	0.3	0.4	2	0.3	0.4

Table 3

(e)

(f)
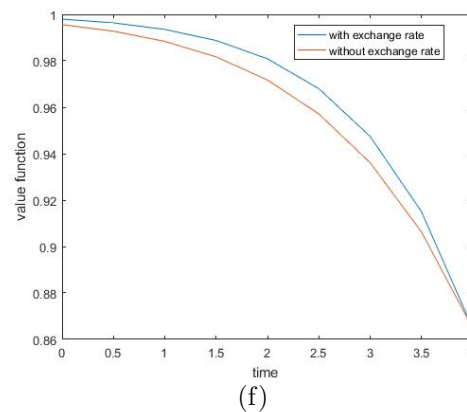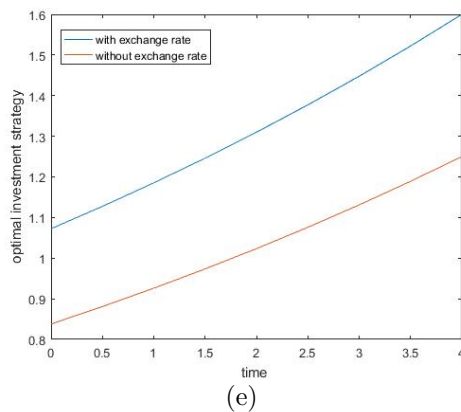
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